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A GEOMETRIC REFORMULATION OF 4-DIMENSIONAL SURGERY*

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Certain topological generalizations of the Schottky groups are described. These act on the three spheres. The question of finding a suitable extension over B^4 is considered an shown to be equivalent to the topological surgery-conjecture. The high-dimensional analogues of these actions are shown to have suitable extensions.

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A major question about topological 4-manifolds is the validity of the 4-dimensional surgery and 5-dimensional s-cobordism 'theorems'. Both are presently known [4] for a class of fundamental groups including those with solvable subgroups of finite index. In both cases there are key (relative) examples, in which the relevant fundamental group is free, in which the 'theorems' apply (respectively) iff they apply in general. The s-cobordism 'theorem' has a completely elementary statement, particularly since we may restrict to the case: $\pi_1 \cong$ free group, where all homotopy equivalences are simple. The statement of the surgery 'theorem' is slightly less tractable to non-specialists. I give here an equivalent reformulation of this problem which is more accessible, attractive for its simplicity, and which appears to invite an insight from another field.

I would like to consider a topological generalization of the Schottky groups, certain free groups on k generators, $k \geqslant 2$, $F_k \subset \mathrm{SO}(n+1, 1)$, well known from the study of Kleinian groups (see [7] for example). The Lie group $\mathrm{SO}(n+1, 1)$ is naturally identified with the (oriented) isometries of hyperbolic $(n+1)$-space H^{n+1} and conformal automorphisms of S^n. Represented in this way, the action of a Schottky group on $(B^{n+1}; H^{n+1} \cong \mathrm{int}\, B^{n+1}, S^n)$, $n \geqslant 2$, may be described topologically as the extension ϕ of the covering transformations

$$\phi_0 \colon F_k \underset{\cong}{\to} \mathrm{Aut}\left(\left(\widetilde{\natural_{k\text{-copies}} S^1 \times D^n}\right)^{\text{universal}} \to \natural_{k\text{-copies}} S^1 \times D^n\right)$$

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to the Freudenthal (i.e., "end point") compactification $\hat{\natural}_k S^1 \times D^n \cong B^{n+1}$. Let us preserve three features from the algebraic model by defining a topological group action $\omega : F_k \to \text{homeo}^+(S^n)$ of a free group F_k, $k \geq 1$, to be *admissible* iff

(1) the limit set $\Lambda_\omega (= \{p \in S^n$ s.t. $\forall q \in S^n$ and $\forall$ neibs. U_p $\omega f(q) \subset U_p$ for infinitely many $f \in F_k\})$ is Cantor set.

(2) ω restricted to $\Omega_\omega = S^n - \Lambda_\omega$ acts totally discontinuously (i.e., for all compact sets $K \subset \Omega$, $\omega f(K) \cap K \neq \emptyset$ for at most finitely many $f \in F_k$), and

(3) the quotient Ω/ω is compact.

Several peculiar admissible actions are discussed in [6]. In the algebraic example, ϕ, Λ_ϕ is a tame Cantor set; this characterizes the Schottky groups topologically among admissible actions for $n \neq 4$ (see Remark).

We say an admissible action ω has an *admissible extension* $\bar{\omega}$ if $\bar{\omega} : F_k \to \text{homeo}^+ B^{n+1}$ satisfies: $\Lambda_{\bar{\omega}} = \Lambda_\omega$, $\bar{\omega}$ acts totally discontinuously on $B^{n+1} - \Lambda_{\bar{\omega}}$ and $\bar{\omega}|_{S^n} = \omega$. We prove:

Theorem 1. *For $n \neq 3$ every admissible action on S^n has an admissible extension to an action on B^{n+1}.*

Theorem 2. *Every admissible action on S^3 has an admissible extension to B^4 iff the topological surgery 'theorem' holds in dimension four.*

(Statement: A normal map of pairs $(M, \partial M) \to (P, \partial P)$ from a topological 4-manifold to a Poincaré pair with a well defined Wall group obstruction $\mathcal{O} \in L_4^s(\pi_1 P)$ is normally cobordant (rel. ∂) to a simple homotopy equivalence iff $\mathcal{O} = 0$.)

The methods of Theorem 1 extend to yield a result not relevant to our main purpose but interesting enough to state. Call an action $\omega : F_k \to \text{homeo}^+(S^n)$, $k \geq 1$, weakly admissible iff (1) Λ_ω is (topologically) zero dimensional, and (2) as above, and condition (3) is dropped.

Addendum to Theorem 1. *For $n \neq 3$ every weakly admissible action on S^n has an admissible extension to an action on B^{n+1}.*

To extend the analogy with Kleinian groups, the hypothesis Theorem 1 corresponds to forbidding parabolics. The addendum says that the conclusion is still true in their presence.

Proof of Theorem 1. It is sufficient to consider the case, $n \geq 4$; the cases $n = 1$ or 2 are classical.

Since $\omega|_{\Omega_\omega}$ is totally discontinuous, $\Omega_\omega \to \Omega_\omega/\omega = M^n$ is a covering projection to a Hausdorf manifold, compact by assumption (3). This covering projection determines a natural map $\pi_1(M^n) \xrightarrow{r_\omega} F_k$. We note that the action $\omega|_{\Omega_\omega}$ is determined up to topological conjugacy by the equivalance class of (M^n, r_ω), where $(M', r') \equiv (M, r)$

iff there is a homeomorphism $h: M' \to M$ with $h_{\#}(\ker r') = \ker r$. Similarly, since $S^n = \bar{\Omega}_\omega$, any equivalence of actions $\omega'|_{\Omega_\omega}' \equiv \omega|_{\Omega_\omega}$ extends over the space of ends to an equivalence $\omega' \equiv \omega$. □

Lemma. *Any M^n, as above, bounds a manifold $K(\pi, 1)$, N^{n+1}, with an isomorphism, $\pi_1(N^{n+1}) \cong F_k$ which identifies $\mathrm{inc}_{\#}: \pi_1(M^n) \to \pi_1(N^{n+1})$ with r_ω.*

Proof. It is an easy exercise in transversality to construct a degree one map $M^n \xrightarrow{\theta} \#_{k\text{-copies}} S^1 \times S^{n-1}$ representing r_ω. Since θ is an integral homology isomorphism, the E_2 term of the relative Atiyah–Hirzebruch spectral sequence vanishes, implying that θ is also an isomorphism on real K-theory. Thus θ is covered by a degree-one normal map.

To see if θ is normally cobordant to the identity, consider the Puppe sequence:

$$\left[\bigvee_k S^1 \vee \Big(\bigvee_k S^{n-1}\Big), G/\mathrm{Top}\right] \xleftarrow{\alpha} \left[\mathop{\#}_k S^1 \times S^{n-1}, G/\mathrm{Top}\right] \xleftarrow{\beta} [S^n, G/\mathrm{top}].$$

An element of the left group consists of k signatures/8 (if $n-1 \equiv 0 \pmod 4$) or Arf invariants (if $n-1 \equiv 2 \pmod 4$) or is trivial (if $n-1$ is odd). These surgery obstructions are defined on the codimension 1 submanifolds $L_i = \theta^{-1}((\text{pt. of } S^1_i) \times S^{n-1})$. We see that these are zero by lifting the submanifolds L_i to the cover Ω_ω, $L_i' \subset \Omega_\omega$. By duality, the submanifold L_i' divides Ω_ω into two (possibly disconnected) pieces. By a Mayer–Vietoris argument, the kernels of the two inclusions are summands A and $B \subset H_{(n-1)/2}(L_i'; \text{Field})$ each of half the total dimension.

Both A and B are maximal null spaces for the intersection form on L_i'. Now, if $n-1 \equiv 0 \pmod 4$ take the field $\cong Q$, since A_Q (say) has half the total dimension, the signature must vanish. (For non-singular forms $|\mathrm{sig}| = \dim - 2 \dim$ (maximal null space)). If $n-1 \equiv 2 \pmod 4$ we must do low dimensional surgery to make the $L_i(\frac{1}{2}(n-1)-1)$-connected. Now, take the field $\cong Z_2$. The surgery obstruction on L_i is the Arf invariant of the natural quadratic intersection form q defined by normal data over the inner product space $H_{(n-1)/2}(L_i'; Z_2)$. As above, q vanishes on A_{Z_2} so $\mathrm{Arf}(q) = 0$. Thus $\alpha(\theta) = 0$ and $\theta \in \text{image } \beta$.

Suppose $n \equiv 0 \pmod 4$, then $\beta^{-1}(\theta) = (\mathrm{sig}\, M^n - \mathrm{sig}(\#_k S^1 \times S^{n-1}))/8$. By Alexander duality $H^*(\Omega_\omega; Z) \cong H(\Omega_\phi; Z)$ so θ is a $Z[F_k]$-homology equivalence. In particular, $K = \ker(r_\omega)$ is a perfect subgroup of $\pi_1(M^n)$. As in Quillen's +-construction, 2 and 3-cells may be attached to M^n (to obtain M^+) so that $\pi_1(M^+) = \pi_1(M^n)/K \equiv F_k$ and the inclusion $M^n \subset M^+$ is a $Z[F_k]$ homology equivalence. The map θ now extends to a map $\bar{\theta}: M^+ \to \cong_k S^1 \times S^{n-1}$ which by Whitehead's theorem is a homotopy equivalence. Defining the signatures from the cohomology rings we have $H^*((\#_k S^1 \times S^{n-1}) \xrightarrow{\cong} H^*(M^+) \xrightarrow{\cong} H^*(M^n)$ implies $\mathrm{sig}\, \#_k S^1 \times S^{n-1} = \mathrm{sig}\, M^n$. So when $n \equiv 0 \pmod 4$ $\beta^{-1}(\theta) = 0$.

Suppose $n \equiv 2 \pmod 4$, then $\beta^{-1}(\theta)$ is the Arf invariant of θ. By [13, Chapter 9] this is equal to the Arf invariant on the bordant surgery problem from a homology sphere to S^n, $\sum^n \xrightarrow{\theta'} S^n$, arising from k framed 1-surgeries on domain and range. Kervaire and Milnor [8] observed that $\mathrm{Arf}(\theta')=0$. Briefly, an equal number of 1 and 2-surgeries bords θ' to a homotopy equivalence. (This is the manifold category version of the +-construction.) In the case $n=\text{odd}$, $\beta^{-1}(\theta)=0$ for trivial reasons. Thus θ is normally cobordant to the identity, $\mathrm{Id}_{\#_k S^1\times S^{n-1}}$.

Gluing an index $=1$ handle body to this normal bordism leads to an $(n+1)$-dimensional surgery problem $\Theta: P^{n+1}\to \natural_k S^1\times D^n$. By Shaneson's thesis [11] the obstruction to normally bording $\Theta(\mathrm{rel}\,\partial)$ to a homotopy equivalence lies in $L^s_{n+1}(F_k)\cong L^s_{n+1}(\{e\})\oplus(\bigoplus_k L^h_n(\{e\}))$ and consists of a top surgery obstruction plus obstructions on θ^{-1} (pt. of $S^1_i\times D^n$) $=L_i$.

By plumbing, the negative of the top obstruction can be realized on a framed manifold bounding a homotopy sphere. Taking boundary connected sum changes the problem Θ to Θ'. The top obstruction on Θ' vanishes, and $\partial\Theta'=\partial\Theta$ except for the normal data. Similarly, the negative of the codimension one obstruction can be realized by a map: $V\to \natural_k S^1\times D^n$ which is a homeomorphism over the boundary. Join the two problems along $\natural_k S^1\times D^{n-1}\subset \partial\natural_k S^1\times D^n$ (this is essentially ∂-connected sum along π_1) to obtain $\Theta': P'\to \natural_k S^1\times D^n$ with vanishing surgery obstruction. Again $\partial\Theta'=\partial\Theta$ except for the normal data. By the surgery theorem P' is normally bordant to the desired manifold N^{n+1}. □

The free group F_k acts as the group of covering automorphisms of the universal covering $\tilde{N}^{n+1}\to N^{n+1}$. We claim that the natural extension to an action of F_k on the Freudenthal compactification $\hat{N}^{n+1}$ is an admissible extension of ω. To verify this, it suffices to show that $\hat{N}^{n+1}$ is homeomorphic to B^{n+1}.

For our limited purposes, we let frontier (Fr) applied to an m-dimensional space denote the subset of points which do not have Euclidian, R^m, neighborhoods. The inclusion $\mathrm{Fr}(\hat{N}^{n+1})\subset \hat{N}^{n+1}$ is 1-u.l.c. (That is, for every x in the frontier and every neighborhood $U_x\subset \hat{N}^{n+1}$ there is a smaller neighborhood V_x so that any loop in $V_x\backslash\mathrm{Fr}$ shrinks in $U_x\backslash\mathrm{Fr}$.) To see this, note that N^{n+1} strong deformation retracts to a wedge $\vee_k S^1$. The deformation retraction f_t has some diameter $d>0$ as measured in $\vee_k S^1$. Lift f_t to obtain $\tilde{f}_t: \tilde{N}^{n+1}\to C$, where C is the universal cover of $\vee_k S^1$. Suppose γ is a loop in $\tilde{N}^{n+1}$ with diameter $=e$ (when measured in C). γ may be deformed into C by a homotopy of diameter $e+2d$ and then shrunk along C to a point with no further increase in diameter. This easily implies that the frontier is 1-u.l.c. Now we may apply the proof of Quinn's mapping cylinder theorem for 1-u.l.c. imbedded ANR's to construct a collaring $S^n\times[0,1]\xrightarrow{\text{homeo}}$ closed neighborhood (Fr N^{n+1}). The proof is based on the controlled h-cobordism theorem (see [9] for $n\geqslant 5$, and [10] for $n=4$). In the codimension one case, the proof proceeds independently on both sides of the ANR; it is not affected by the absence of an ambient manifold in which the ANR is embedded. The collaring shows that $\hat{N}^{n+1}$ is a manifold with boundary and by the high-dimensional Poincaré conjecture $\hat{N}^{n+1}\cong B^{n+1}$. Alternate proofs that $N\cong B^{n+1}$ based on ideas of Cannon or Daverman may be given. □

Remark. If the domain of discontinuity Ω_ω is simply connected then the map $r_\omega: M^n \to \#_k S^1 \times S^{n-1}$ is a homotopy equivalence. Observe that the manifold constructed in Theorem 1 with a neighborhood of its 1-spine deleted, $N^{n+1} \setminus \dot{\mathcal{N}}(\vee_k S^1)$, is an h-cobordism. When $n \geq 5$, it follows that M^n is homeomorphic to $\#_k S^1 \times S^{n-1}$. (When $n = 3$ we may recognize M^n above, not by the h-cobordism theorem, but by using the sphere theorem and the fact that the imbedding of its universal cover in S^3 rules out the existence of fake 3-cells.) From the remarks at the beginning of the proof of Theorem 1, we conclude that if Ω_ω is simply connected and $n \neq 4$ the action ω is topologically equivalent to ϕ.

Proof of Addendum (sketch). Using the density of orbits in Λ_ω one may check that Λ_ω is a Cantor set, two points, or one point. The quotient $\Omega_\omega / \omega = M$ will be noncompact unless ω is actually admissible. It is possible to construct a proper map f from M to an appropriate model space P. This has the form $P \cong \#_k(S^1 \times S^{n-1}) \setminus \amalg \vee S^1$, where the deleted set is a disjoint union of tame wedges of circles (a total of l circles, $l \leq k$, are involved and they correspond to part of a generating set of F_k under the projection r_ω.) The model space P is chosen by first cutting M open dual to the generators of F_k and then looking for the model which when similarly cut open has the corresponding pattern of ends, non-compact boundary components, and compact boundary components. A proper degree-one map between the cut open objects is easily found; glue back to form f. The cases $n < 3$ are handled by example; assume $n \geq 5$.

One checks that f is a $Z[F_k]$-homology equivalence. Using arguments similar to those in the proof of Theorem 1, but now in the context of the surgery on proper maps (this theory is due to Taylor [12]), f may be covered by a bundle map so that (f, b) is normally cobordant to id_P. Furthermore the normal cobordism W can be chosen to carry the zero surgery obstruction and thus may be taken to be a proper $Z[F_k]$-homology-cobordism from M to P. Since W has dimension at least 5 we may modify W by a $+$-construction to assure that $\pi_1(W) \cong F_k$ and then by a 'proper $+$-construction' (i.e., proper collection of 1-surgeries and 2-surgeries) so that the fundamental group of each end of W is stable and free.

The model P contains a natural basis for $H_{n-1}(P, Z)$ consisting of imbedded $(n-1)$-spheres. Attach n-handles to these spheres to form W^+ and delete $(\partial W^+ \setminus M)$ to form X. Let Y be the universal cover of X and Z the Freudenthal compactification of Y. Our care with the fundamental groups of the ends of W makes Z an ANR with 1-u.l.c. frontier. One checks, as in Theorem 1, that Z is homeomorphic to B^{n+1}. The covering, translations of $(Y \to Z)$ extended to Z constitute the admissible extension. $\square$

Proof of Theorem 2. First, suppose the surgery theorem (topological category) holds in dimension = four. As in the proof of Theorem 1, a surgery problem $\Theta: P \to \natural_k S^1 \times D^3$ with $\partial P = M^3$ can be constructed. Because of the existence [3] of an almost-framed index = 8 manifold, $|E_8|$, (P, Θ) may be altered, as in Theorem 1, to arrange $[\Theta]=0\in L_4^s(F_k)\equiv Z$. Now complete surgery to construct N^4. Finally, we must see that $\hat{N}^4\cong B^4$. For this, one uses the same destabilization argument [10] that yields the local flatness criteria for three manifolds imbedded in a 4-manifold. Briefly, one constructs a product collar on $\mathrm{Fr}(\hat{\hat{N}}^4\times R)$ inside $\hat{N}^4\times R$ using the controlled 5-dimensional h-cobordism theorem. This collar contains disjointly an out-of-level-collar $\mathrm{Fr}(\hat{N}^4)\times(0,1]$ and an open neighborhood U of the frontier in $\hat{N}^4\times 0$. Between these is another controlled 5-dimensional h-cobordism; another application of the same theorem produces an imbedding of $\mathrm{Fr}(\hat{N}^4)\times(1-\varepsilon,1)$ into U which completes to a collaring of $\mathrm{Fr}(\hat{N})\times 0$ in the 0-level. This establishes that that $\hat{N}^4$ is a topological manifold; the 4-dimensional Poincaré conjecture [3] implies $\hat{N}^4\cong B^4$.

Second, we suppose that every admissible ω acting on S^3 has an admissible extension $\bar{\omega}$; from this we deduce the surgery theorem.

It is known [2, 4, 5] that the general (topological 4-dimensional) surgery theorem is implied by special cases originally called 'atomic surgery problems' because others could be decomposed into these. Furthermore, the five-dimensional normal bordism can always be constructed after the fact given the appropriate 'solution' 4-manifold so that the issue is simply whether, for a certain class of 3-manifolds, $\mathcal{M}$, the appropriate bounding 4-manifolds can be constructed. $\mathcal{M}=\{M_i^3\}$ may be taken to consist of 3-manifolds which result from 0-framed surgery on a link $L_i\in\{L_i\}=\mathcal{L}$. Each L_i is a (possibly) ramified Whitehead-double of a (possibly) ramified Bing-double of the Hopf link.

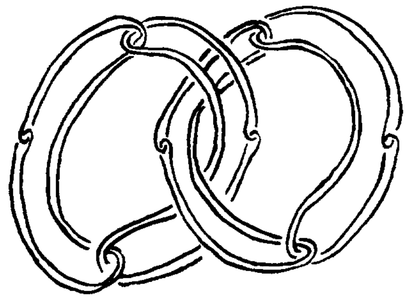

Fig. 1.

Since L_i is a boundary link in an obvious way there is a well defined epimorphism $\rho_i:\pi_1(M_i^3)\to F_{k_i}$. Being an 'appropriate' bounding N_i^4 means:

(1) $\partial N_i^4=M_i^3$,

(2) the inclusion of a wedge of meridinal circles to L_i determines a homotopy equivalence $\bigvee_k S^1 \xrightarrow{\simeq} N_i^4$, and

(3) $\pi_1(M_1^3)\to\pi_1(N_i^4)$ is onto with kernel $K=\ker\rho_i$.

It will be seen that for all i, $\hat{M}_i^{\rho_i}$ is homeomorphic to S^3. Granting this let ω_i be the (completed) covering action and let $\bar{\omega}_i$ be the hypothesized extension to B^4. Define $N_i=(B^4-\Lambda_{\omega_i})/\omega_i$. Properties (1) and (2) above are immediate. The fundamental group $\pi_1(N_i^4)$ is identified with the covering translations of $\Omega_{\omega_i}\to M_i^3$. This determines an action of $\pi_1(N_i^4)$ on $\pi_1(\Omega_{\omega_i})$ with $\pi_1(M_i^3)$ being the resultant semi-direct product $\pi_1(\Omega_{\omega_i}) \rtimes \pi_1(N_i^4)$. Thus we have:

$$\begin{array}{ccccccccc}
0 & \to & K & \to & \pi_1(M_i^3) & \to & \pi_1(N_i^4) & \to & 0 \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
0 & \to & \pi_1(\Omega_{\omega_i}) & \to & \pi_1(\Omega_{\omega_i} \rtimes \pi_1(N_i^4) & \to & \pi_1(N_i^4) & \to & 0 \\
 & & \wr\| & & & & & & \\
 & & \ker \rho_i & & & & & &
\end{array}$$

establishing property (3).

Finally, we show that for every $L_i \in \mathscr{L}$, $\hat{M}_i^{\rho_i}$ is homeomorphic to S^3. The following fact is helpful:

Identity on the zero framed surgery of Whitehead doubles: Let $J \subset S^3$ be a smooth link of j components. Let $\mathrm{Wh}(J)$ denote a Whitehead double where each component of J is replaced by an untwisted + or − satellite based on

 $\subset S^1 \times D^2$ or  $\subset S^1 \times D^2$

respectively. Write the result of zero framed surgery, $\mathscr{S}(\mathrm{Wh}(J))$. Let $D(J)$ denote the link of $2j$ components obtained by pushing zero-linking parallels off the components of J. Let Q be the 3-manifold $(S^3 - \mathscr{N}(D(J))/\partial-$identifications) where parallel boundary components are identified in pairs by:

(1) case+ :meridian → longitude and longitude → meridian, or

(2) case−: meridian → −longitude and longitude → −meridian.

Claim: If the two cases, ±, of identification correspond with two types, ±, of satellites we will have: $\mathscr{S}(\mathrm{Wh}(J)) \cong P$. Furthermore, both $\pi_1(\mathrm{Wh}(J))$ and $\pi_1 P$ map naturally onto F. (Use the obvious representation of $\mathrm{Wh}(J)$ as a boundary link in the first case, and intersection with the identified tori in the latter.) The above isomorphism induces a commutative diagram:

$$\begin{tikzcd}
\pi_1(\mathrm{Wh}(J)) \arrow[r, phantom, "\cong"] \arrow[dr] & \pi_1 Q \arrow[d] \\
 & F_j
\end{tikzcd}$$

Verification: Representing J schematically: $\square$, in Kirby calculus notation [3] we have

$\mathscr{S}(\mathrm{Wh}(J)) =$ 

(note that the clasp is drawn ambiguously). Using the interpretation of $\mathscr{S}(\mathrm{Wh}(J))$

in terms of kinky handles attached to $\mathcal{S}(J)$ one sees:

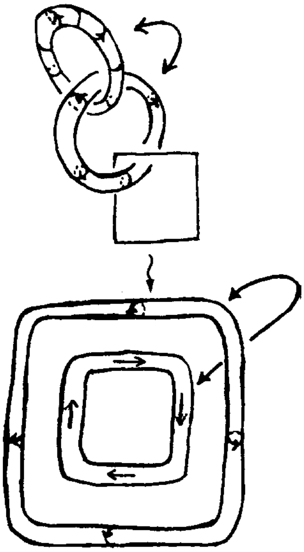

Fig. 2.

Thus some complement of the form: $S^3 - \mathring{\mathcal{N}}D(\text{Bing}(\text{Hopf}))$ (=the 3-sphere minus the interior of a tubular neighborhood of the double of a (possibly) ramified Bing double [5] of the Hopf link) is a fundamental domain Δ_i for the action ω_i on $\tilde{M}_i^{\rho_i}$. The gluing data allows the attachment of adjacent domains to be performed within S^3. Up to the ± 1 ambiguity of the gluing matrix we may draw (in part) an imbedding of $\tilde{M}_i^{\rho_i}$ in S^3. In the case with no ramification, the picture for the two adjacent fundamental domains is sketched below.

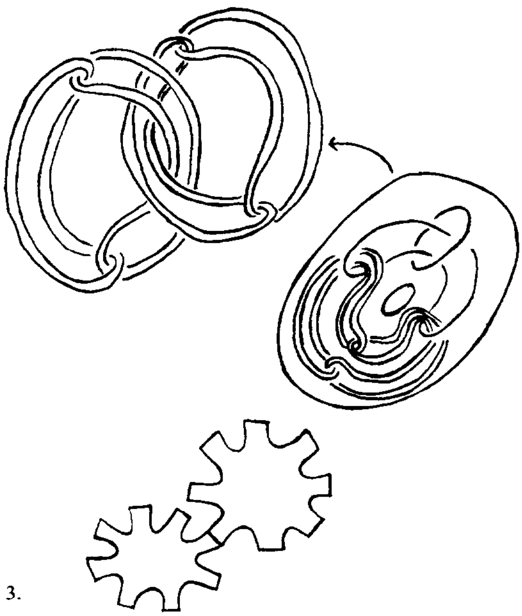

Fig. 3.

Consider all solid tori in S^3 which are bounded by translates of the boundary components of Δ_i but which do not contain Δ_i. Let $\bar{\mathscr{D}} \subset S^3$ be the set of points in infinitely many such solid tori. Let $\mathscr{D}_i$ be the set of connected components of $\bar{\mathscr{D}}_i$. The Freudenthal compactification $\hat{M}_i^{\rho_i}$ is homeomorphic to the space obtained by identifying each element of $\mathscr{D}_i$ to a point, $S^3/\mathscr{D}_i$.

The nesting of solid tori is essentially no worse than in the classical example examined by Bing during his investigation of the doubled Alexander-horned ball [1]. It is an easy exercise to adapt Bing's argument to shrink $\mathscr{D}_i$. Thus $\hat{M}_i^{\rho_i} \cong S^3/\mathscr{D}_i \cong S^3$. □

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